Solutions to tutorial exercises for stochastic processes

T1. The process $(M+N)_t$ is increasing and right-continuous, since M_t and N_t are increasing and right-continuous. Furthermore

$$(M+N)_t - (M+N)_s = M_t - M_s + N_t - N_s \sim \mathrm{POI}\big((\lambda+\mu)(t-s)\big),$$

since $M_t - M_s$ and $N_t - N_s$ are independent and Poisson distributed with parameter $\lambda(t-s)$ and $\mu(t-s)$ respectively. It remains to show that $(M+N)_t$ has steps of size 1 almost surely. Construct the process M'_t by placing $X_i \sim \mathrm{POI}(\lambda)$ points, $x_1^i, \dots, x_k^i$, uniformly at random in the interval $[i, i+1)$, so that $M'_t \overset{d}{=} M_t$. Similarly construct $N'_t \overset{d}{=} N_t$ by placing the points $y_1^i, \dots, y_l^i$ in the interval $[i, i+1)$. Then $(M'+N')_t \overset{d}{=} (M+N)_t$. Suppose $(M'+N')_t$ has a jump of size 2. Then there exists an interval $[i, i+1)$ such that $x_v^i = y_w^i$ for some $v, w \in \mathbb{N}$. Now,

$$\begin{aligned}
\mathbb{P}\big((M'+N')_t \text{ has jump of size } 2\big) &\le \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{v=1}^{k}\sum_{w=1}^{l} \mathbb{P}(X_i = k, Y_i = l, x_v^i = y_w^i) \\
&= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{v=1}^{k}\sum_{w=1}^{l} \mathbb{P}(X_i = k)\mathbb{P}(Y_i = l)\mathbb{P}(x_v^i = y_w^i) \\
&= 0,
\end{aligned}$$

since $\mathbb{P}(x_v^i = y_w^i) = 0$. So $(M'+N')_t$ has steps of size 1 almost surely and thus $(M+N)_t$ as well.

T2. Let $O = (O_{ij})$. The vector $Y := OX$ has a multivariate Gaussian distribution: for $a_1, \dots, a_n \in \mathbb{R}$ we have

$$\sum_{j=1}^n a_j Y_j = \sum_{j=1}^n a_j \sum_{i=1}^n O_{ij} X_i = \sum_{i=1}^n b_i X_i,$$

for some $b_1, \dots, b_n \in \mathbb{R}$, so that this sum is Gaussian. Furthermore for $1 \le i \le n$ and $1 \le j \le n$ we have

$$\mathbb{E}[Y_j] = \mathbb{E}\left[\sum_{i=1}^n O_{ij} X_i\right] = 0,$$

and

$$\mathrm{Cov}(Y_i, Y_j) = \mathrm{Cov}\left(\sum_{k=1}^n O_{ki} X_k, \sum_{k=1}^n O_{kj} X_k\right) = \sum_{k=1}^n O_{ki} O_{kj} \mathrm{Cov}(X_k, X_k) = \mathbb{1}_{\{i=j\}},$$

since O is an orthogonal matrix. Since the distribution of a multivariate Gaussian random variable is determined by its expectation and its covariance matrix, it follows that

$$OX \stackrel{d}{=} X.$$

T3. Note that the company can only go bankrupt at an arrival time of a claim. We define $\psi_k(u)$ as the probability that the company goes bankrupt at or before the kth claim. We have $\psi_k \to \psi(u)$ as $k \to \infty$, so it suffices to show that $\psi_k(u) \leq \exp(-Ru)$ for all $k \in \mathbb{N}$. We define $\psi_0(u) = 0$, so that $\psi_0(u) \leq \exp(-Ru)$. We now use induction on k: we suppose $\psi_{k-1}(u) \leq \exp(-Ru)$. Conditioning on the time and the size of the first claim gives

$$\begin{aligned}
\psi_k(u) &= \int_0^\infty \int_0^\infty \psi_{k-1}(u + ct - x)\mu(\mathrm{d}x)\lambda e^{-\lambda t}\mathrm{d}t \\
&\leq \int_0^\infty \int_0^\infty \exp\big(-R(u + ct - x)\big)\mu(\mathrm{d}x)\lambda e^{-\lambda t}\mathrm{d}t \\
&= e^{-Ru} \int_0^\infty e^{Rx}\mu(\mathrm{d}x) \int_0^\infty \lambda \exp\big(-t(cR + \lambda)\big)\mathrm{d}t \\
&= e^{-Ru} \frac{\lambda}{cR + \lambda} m_X(R) \\
&= e^{-Ru}.
\end{aligned}$$

(Formally we make use of the strong Markov property in the first equation above, but this will only be introduced later in the course.)